MTH 166

Lecture-19

Separation of variables solution of Partial Differential Equations

Topic:

Solution of Partial Differential Equations

Learning Outcomes:

1. To solve first order PDE using separation of variables solution.

2. To find nature of Heat, Wave and Laplace equation.

Find the separation of variables solution of following PDE:

Problem 1. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$

Solution. The given equation is: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ (1)

Let solution be: $u(x, y) = XY$ (2) where $X = f(x), Y = g(y)$

$\Rightarrow \frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$

Equation (1) becomes: $X'Y = XY'$

$\Rightarrow \frac{X'}{X} = \frac{Y'}{Y} = k$ (Say)

Taking these pairs one by one

$\Rightarrow \frac{X'}{X} = k$

$$\Rightarrow \int \frac{X'}{X} dx = k \int dx$$

$$\Rightarrow \log X = kx + c_1$$

$$\Rightarrow X = e^{(kx+c_1)}$$

also $\frac{Y'}{Y} = k$

$$\Rightarrow \int \frac{Y'}{Y} dy = k \int dy$$

$$\Rightarrow \log Y = ky + c_2$$

$$\Rightarrow Y = e^{(ky+c_2)}$$

Required solution is: $u(x, y) = XY = e^{(kx+c_1)} e^{(ky+c_2)} = e^{(c_1+c_2)} e^{k(x+y)}$

$\Rightarrow u(x, y) = Ae^{k(x+y)}$ **Answer.** $(A = e^{(c_1+c_2)})$

Find the separation of variables solution of following PDE:

Problem 2. $4\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial x}} + \mathbf{3}\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial y}} = \mathbf{0}$

Solution. The given equation is: $4\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 0$ (1)

Let solution be: $u(x, y) = XY$ (2) where $X = f(x), Y = g(y)$

$\Rightarrow \frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$

Equation (1) becomes: $4X'Y + 3XY' = 0 \quad \Rightarrow 4X'Y = -3XY'$

$\Rightarrow 4\frac{X'}{X} = -3\frac{Y'}{Y} = k$ (Say)

Taking these pairs one by one

$\Rightarrow 4\frac{X'}{X} = k$

$$\Rightarrow \int \frac{X'}{X}\,dx = \frac{k}{4}\int dx$$

$$\Rightarrow \log X = \frac{k}{4}x + c_1$$

$$\Rightarrow X = e^{\left(\frac{k}{4}x + c_1\right)}$$

also $\frac{Y'}{Y} = k$

$$\Rightarrow \int \frac{Y'}{Y}\,dy = -\frac{k}{3}\int dy$$

$$\Rightarrow \log Y = -\frac{k}{3}y + c_2$$

$$\Rightarrow Y = e^{\left(-\frac{k}{3}y + c_2\right)}$$

Required solution is: $u(x,y) = XY = e^{\left(\frac{k}{4}x + c_1\right)}e^{\left(-\frac{k}{3}y + c_2\right)} = e^{(c_1 + c_2)}e^{k(\frac{1}{4}x - \frac{1}{3}y)}$

$\Rightarrow u(x,y) = Ae^{\frac{k}{12}(3x - 4y)}$ **Answer.** $\left(A = e^{(c_1 + c_2)}\right)$

<u>Find the separation of variables solution of following PDE:</u>

<u>Problem 3.</u> $\quad y\frac{\partial u}{\partial x} + \boldsymbol{x}\frac{\partial u}{\partial y} = \mathbf{0}$

<u>Solution.</u> The given equation is: $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0 \qquad (1)$

Let solution be: $u(x, y) = XY \qquad (2) \qquad$ where $X = f(x), Y = g(y)$

$\Rightarrow \frac{\partial u}{\partial x} = X'Y$ and $\frac{\partial u}{\partial y} = XY'$

Equation (1) becomes: $yX'Y + xXY' = 0 \quad \Rightarrow yX'Y = -xXY'$

$\Rightarrow \frac{1}{x}\frac{X'}{X} = -\frac{1}{y}\frac{Y'}{Y} = k$ (Say)

Taking these pairs one by one

$\Rightarrow \frac{1}{x}\frac{X'}{X} = k$

$$\Rightarrow \int \frac{X'}{X} dx = k \int x dx$$

$$\Rightarrow \log X = k \frac{x^2}{2} + c_1$$

$$\Rightarrow X = e^{\left(k\frac{x^2}{2}+c_1\right)}$$

also $-\frac{1}{y} \frac{Y'}{Y} = k$

$$\Rightarrow \int \frac{Y'}{Y} dy = -k \int y dy$$

$$\Rightarrow \log Y = -k \frac{y^2}{2} + c_2$$

$$\Rightarrow Y = e^{\left(-k\frac{y^2}{2}+c_2\right)}$$

Required solution is: $u(x, y) = XY = e^{\left(k\frac{x^2}{2}+c_1\right)} e^{\left(-k\frac{y^2}{2}+c_2\right)} = e^{(c_1+c_2)} e^{k(\frac{x^2}{2}-\frac{y^2}{2})}$

$\Rightarrow u(x, y) = Ae^{\frac{k}{2}(x^2-y^2)}$ **Answer.** $(A = e^{(c_1+c_2)})$

Classification/Nature of Partial Differential Equations:

Let us consider a general second order homogeneous PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\,\frac{\partial^2 u}{\partial x \partial y} + \mathrm{C}\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + Fu = 0 \quad \text{where} \quad u = f(x, y)$$

or

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0 \qquad (1)$$

Equation (1) will be:

1. Hyperbolic iff $(B^2 - 4AC) > 0$

2. Parabolic iff $(B^2 - 4AC) = 0$

3. Elliptic iff $(B^2 - 4AC) < 0$

<u>Note:</u> Coefficients of second order partial derivatives only decide nature of PDE.

Classify the following PDE as Hyperbolic, Parabolic or Elliptic:

Problem 1. $\boldsymbol{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$ **[2D-Laplace Equation]**

Solution. Given equ. is: $u_{xx} + u_{yy} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = 1, B = 0, C = 1$

Here $(B^2 - 4AC) = (0)^2 - 4(1)(1) = -4 < 0$

So, **Laplace** equation (1) is **Elliptic** in nature.

Problem 2. $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ **[1D-Wave Equation]; c^2 is Diffusivity constant.**

Solution. Given equ. is: $c^2\, u_{xx} - u_{tt} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = 0$

We have $A = c^2, B = 0, C = -1$

Here $(B^2 - 4AC) = (0)^2 - 4(c^2)(-1) = 4c^2 > 0$

So, **Wave** equation (1) is **Hyperbolic** in nature.

Problem 3. $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ **[1D-Heat Equation]; c^2 is Diffusivity constant.**

Solution. Given equ. is: $c^2\, u_{xx} - u_t = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu = 0$

We have $A = c^2, B = 0, C = 0$

Here $(B^2 - 4AC) = (0)^2 - 4(c^2)(0) = 0$

So, **Heat** equation (1) is **Parabolic** in nature.

Problem 4. $\mathbf{(1-y)\frac{\partial^2 u}{\partial x^2} + 2x\frac{\partial^2 u}{\partial x \partial y} + (1+y)\frac{\partial^2 u}{\partial y^2} = 0}$

Solution. Given equ. is: $(1-y)u_{xx}+2xyu_{xy} + (1+y)u_{yy} = 0$ (1)

Comparing with: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$

We have $A = (1-y), B = 2x, C = (1+y)$

Here $(B^2 - 4AC) = (2x)^2 - 4(1-y)(1+y) = 4(x^2 + y^2 - 1)$

So, equation (1) is:

Hyperbolic iff $(x^2 + y^2 - 1) > 0$

Parabolic iff $(x^2 + y^2 - 1) = 0$

Elliptic iff $(x^2 + y^2 - 1) < 0$

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